Genetic Algorithms

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Agenda

GA a simulation by hand

For a function $f(x) = x^2$ where $x \in [0, 31]$ a population of 4 strings.

String No.	Initial Population (Randomly Generated)	x Value (Unsigned Integer)	$f(x)$ x^2	pselect, $\frac{f_i}{\Sigma f}$	Expected count $\frac{f_i}{\bar{f}}$	Actual Count from Roulette Wheel
1	0 1 1 0 1	13	169	0.14	0.58	1
2	1 1 0 0 0	24	576	0.49	1.97	2
3	0 1 0 0 0	8	64	0.06	0.22	0
4	1 0 0 1 1	19	361	0.31	1.23	1
Sum			1170	1.00	4.00	4.0
Average			293	0.25	1.00	1.0
Max			576	0.49	1.97	2.0

GA a simulation by hand

Reproduction, and Crossover with no mutation.

Mating Pool after Reproduction (Cross Site Shown)	Mate (Randomly Selected)	Crossover Site (Randomly Selected)	New Population	x Value	$f(x)$ x^2
0 1 1 0 | 1	2	4	0 1 1 0 0	12	144
1 1 0 0 | 0	1	4	1 1 0 0 1	25	625
1 1 | 0 0 0	4	2	1 1 0 1 1	27	729
1 0 | 0 1 1	3	2	1 0 0 0 0	16	256
					1754
					439
					729

- The population average fitness improved from 239 to 439.
- The maximum fitness also improved from 576 to 729

Outline

Similarity Schema

What information is contained in this population to guide directed search for improvement? (i.e., causal relation similarity/fitness)

String	Fitness
01101	169
11000	576
01000	64
10011	361

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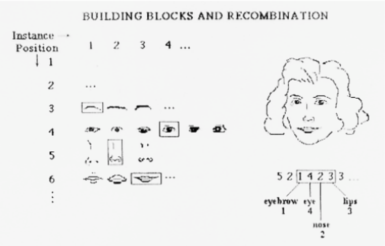
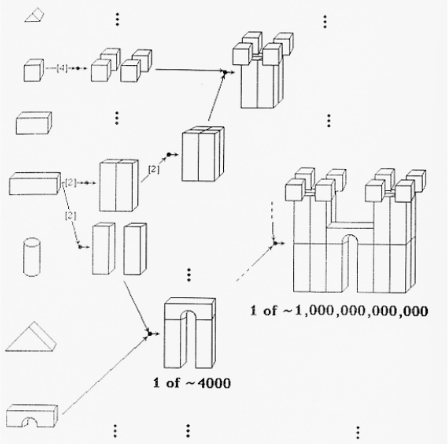
- Strings starts with 1 on the left seems to be better.
- Similar strings fall under the same schema (i.e., similarity template)
- Study of schemata proves the power of GA.

Similarity Schema





The Power of Building Blocks

Similarity templates (Schemeta)

Schema - Holland (1975)

Schema is a template that identifies a subset of strings with similarities at certain string positions.

E.g.

0*1100*

Note

We can think of it as a pattern matching device: a schema matches a particular string if at every location in the schema a 1 matches a 1 in the string, or a 0 matches a 0, or a * matches either.

E.g. For a binary alphabet {**0, 1**}, we motivate a schema by appending a special symbol *****, or **dont care symbol**, producing a ternary alphabet {**0, 1, ***} that allows us to build schemata.

Notation: String, Population

Consider strings to be constructed over the binary alphabet

$$V=\{0, 1\}$$

- **Strings** as capital letters
- **Individual characters** by lowercase letters subscripted by their position.

Example

A = 0111000 may be represented symbolically as:
$A = a_1 a_2 a_3 a_4 a_5 a_6 a_7$

Example

a_i represents a **gene** (binary feature or detector)
a_i **value** represents an **allele**
A(t)represents a population of strings at time (or generation) **t**

Notation: Schema

Consider a schema H taken from the three-letter alphabet:

$$V=\{ 0, 1, * \};$$

* **asterisk** is a **dont care symbol** which matches either a 0 or a 1 at a particular position.

Schema Matching

A **bit string** matches a particular **schemata** if that bit string can be constructed from the schemata by replacing the "*" symbol with the appropriate bit value.

Example

H = *11*0**

String A = 0111000

String A is an example of the schema H because the string alleles a_i match schema positions h_i at the fixed positions 2, 3 and 5.

Outline

Schema Properties

Defining Length of Schema

$\delta(H)$ is the distance between the first and last fixed string position

Ex.

- $H = 011 * 1 * *$
- $\delta(H) = 5 - 1 = 4$

Order of Schema:

$o(H)$ is the number of fixed positions present in the template

- $H = 0 * * * * * *$
- $\delta(H) = 0$ because there is only one fixed position
- $o(H) = 1$

Schema Properties

Note

Schemata and their properties serve as notational devices for rigorously discussing and classifying string similarities.

Note

They provide the basic means for analyzing the **net effect of reproduction and genetic operators** on the building blocks contained within the population.

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How many possible schemata for binary string population of length ($\ell = 5$)

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- Each character can be from $\{0, 1, *\}$
- Thus, we have $3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$ possible schemata
- Alphabet of cardinality k has $(k + 1)^\ell$ possible schemata

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How many schemata are usefully processed?
(is there a lower bound linked to the population size n)?
Holland $o(n^3)$.

Outline

Effect of Reproduction on Schemata

Suppose at time **t**, there are **m** examples of a particular schema **H** in population **A(t)**

$$m = m(H, t)$$

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During reproduction, a string A_i gets copied according to its fitness with probability $p_i = \frac{f_i}{\sum f_i}$

$$m(H, t+1) = m(H, t) \times n \times \frac{f(H)}{\sum f_i}$$

f(H) is the **average fitness** of the strings representing schema **H** at time **t**.

Effect of Reproduction on Schemata

we may write the reproductive schema growth equation as:

$$m(H, t+1) = m(H, t) \times n \times \frac{f(H)}{\sum f_i}$$

Simplification

- If we recognize that the average fitness of the entire population as $\overline{f} = \frac{\sum f_i}{n}$
- we may express the reproductive schema growth equation as:

$$m(H, t+1) = m(H, t) \times \frac{f(H)}{\overline{f}}$$

Effect of Reproduction on Schemata

Reproductive schema growth equation:

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- A particular schema grows as the ratio of the average fitness of the schema to the average fitness of the population

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- Schemata with **fitness values** above the population average will receive an increasing number of samples in the next generation.
- Schemata with **fitness values** below the population average will receive a decreasing number of samples.
- **All the schemata** in a population grow or decay according to their schema averages under the operation of reproduction alone.

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$$m(H, t+1) = m(H, t) \times \frac{\overline{f}+c\overline{f}}{\overline{f}} = m(H, t) \times (1+c)$$

Starting at **t=0**, and assuming a stationary value of **c**, we obtain the equation:

$$m(H, t+1) = m(H, 0) \times (1+c)^t$$

Note

Reproduction allocates exponentially increasing (decreasing) numbers of trials to above (below) average schema.

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- Many different schemata are sampled in **parallel** according to the same rule through the use of **n** simple reproduction operations.
- However, **reproduction does not promote exploration of new regions of the search space**.
- This is where **crossover** steps in

Effect of Crossover on Schemata

Consider a particular string of length $\ell = 7$ and two representative schemata within that string:

$$A = 0111000$$
$$H_1 = *1****0$$
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Recall: Crossover Operation

- crossover proceeds with the random selection of a mate;
- Random selection of a crossover site
- The exchange of substrings from the beginning of the string to the crossover site inclusively with the corresponding substring of the chosen mate.

Effect of Crossover on Schemata

Assuming that we have the following randomly chosen **crossover site: 3**

$$A = 011|1000$$
$$H_1 = *1*|***0$$
$$H_2 = ***|10**$$

- H_1 is **destroyed**. Defining length $= 5$
- H_2 will **survive**. Defining length $= 1$

Note

H_1 is less likely to survive crossover than schema H_2 because on average the crossover site is more likely to fall between the extreme fixed positions.

Lower Bound on Crossover Survival Probability

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- $p_d(H_1) = \frac{\delta(H_1)}{\ell - 1} = \frac{5}{6}$
- $p_d(H_2) = \frac{\delta(H_2)}{\ell - 1} = \frac{1}{6}$
- $p_s(H_1) = 1 - p_d(H_1) = \frac{1}{6}$
- $p_s(H_2) = 1 - p_d(H_2) = \frac{5}{6}$

To generalize, a schema survives when the cross over site falls outside the defining length. The survival probability under simple crossover is $p_s(H)$

$$p_s(H) = 1 - \frac{\delta(H)}{\ell - 1}$$

Lower Bound on Crossover Survival Probability

If we consider the probability of performing a crossover operation to be p_c

$$p_s(H) = 1 - p_c\left(\frac{\delta(H)}{\ell-1}\right)$$

- Independence is assumed between the two event (crossover and schemata destruction)

Combined Effect of Reproduction and Crossover

Assuming independence of the **reproduction** and **crossover** operations.

$$m(H, t+1) = m(H, 0) \times (1+c)^t \times \left[1 - p_c\left(\frac{\delta(H)}{\ell - 1}\right)\right]$$

Schema **H** grows or decays depending upon a multiplication factor.

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Note

Clearly, those schemata with both above-average observed performance and short defining lengths are going to be sampled at exponentially increasing rates.

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For small values of p_m ($p_m << 1$), we can write:

$$(1 - o(H)p_m)$$

Fundamental Theorem of Genetic Algorithms

$$m(H, t+1) \geq m(H, t) \times \frac{f(H)}{\overline{f}} \times \left[1 - p_c\left(\frac{\delta(H)}{\ell - 1}\right) - o(H)p_m\right]$$

- $m(H, t+1)$ Expected Count of Schema H at time (t+1)
- $m(H, t)$ Expected Count of Schema H at time (t)
- $\frac{f(H)}{\overline{f}}$ ratio of schema fitness to the total fitness
- $\left[1 - p_c\left(\frac{\delta(H)}{\ell - 1}\right) - o(H)p_m\right]$ Survival probability

Who shall live and who shall die?

Short, low-order, above-average schemata are given exponentially increasing trials in subsequent generations (building blocks)

Outline

Schema processing by hand

Let us observe how the GA processes schemata not individual strings-within the population

Let us consider three particular schemata, H_1, H_2 and H_3 Where

- $H_1 = 1****$
- $H_2 = *10**$
- $H_3 = 1***0$

Observe the effect of reproduction, crossover, and mutation.

Hand Calculations

String Processing

String No.	Initial Population (Randomly Generated)	x Value (Unsigned Integer)	$f(x)$ x^2	pselect, $\frac{f_i}{\Sigma f}$	Expected count $\frac{f_i}{\bar{f}}$	Actual Count from Roulette Wheel
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Schema Processing

		Before Reproduction	
		String Representatives	Schema Average Fitness $f(H)$
H_1	1 * * * *	2,4	469
H_2	* 1 0 * *	2,3	320
H_3	1 * * * 0	2	576

Hand Calculations

String Processing

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Sum					1754
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Max					729

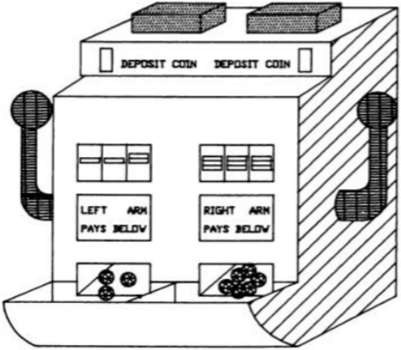
Schema Processing

After Reproduction			After All Operators		
Expected Count	Actual Count	String Representatives	Expected Count	Actual Count	String Representatives
3.20	3	2,3,4	3.20	3	2,3,4
2.18	2	2,3	1.64	2	2,3
1.97	2	2,3	0.0	1	4

Outline

Two Armed Bandit Problem

- Suppose a two armed slot machine where one arm pays a reward μ_1 and variance σ_1 and the other arm pays μ_2 and variance σ_2.
- where $\mu_1 \geq \mu_2$ Which arm should we play?



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- We can give each arm a try or some trials then play with the arm that pay more.
- This is known as a trade-off between the exploration for knowledge and the exploitation of that knowledge.
- Suppose we have a total of N trials to allocate among the two arms. We first allocate an equal number of trials n ($2n \leq N$) trials to each of the two arms.

Two Armed Bandit Problem

we can calculate the expected loss:

$$L(N, n) = |\mu_1 - \mu_2|.\left[(N - n)q + n(1 - q)\right]$$
$$\text{Where } q \approx \frac{1}{\sqrt{2.\pi}} \frac{e^{-x^2/2}}{x} \text{ and } x = \frac{\mu_1 - \mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}.\sqrt{n}$$

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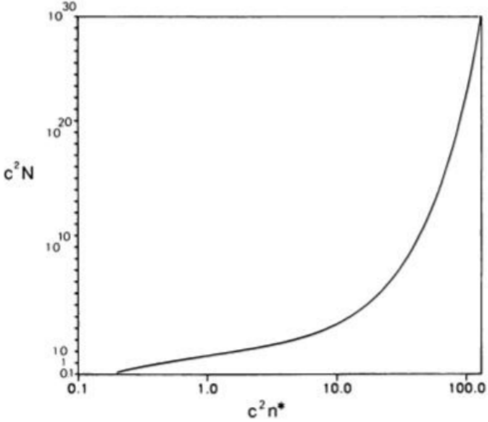
two sources of loss are associated with the procedure.

- The first loss is a result of issuing n trials to the wrong arm during the experiment.
- The second is a result of choosing the arm associated with the lower payoff even after performing the experiment.

if N, μ_1, μ_2, σ_1, σ_2 are known, how to get optimal n^*

Holland (1975) has performed calculations that show how trials should be allocated between the two arms to minimize expected losses.

- $n^* \approx b^2 ln\left[\frac{N^2}{8\pi b^4 lnN^2}\right]$
- $N \approx \sqrt{8\pi b^4 lnN^2} e^{n*/2b^2}$



We should give slightly more than exponentially increasing trials to the observed best arm. The same conclusion apply to the k-armed bandit.

GA and K-Armed Bandit Problem

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E.x.

There are eight $= 2^3$ competing schemata over the three positions 2, 3, and 5

```
* 0 0 * 0 * *
* 0 0 * 1 * *
* 0 1 * 0 * *
* 0 1 * 1 * *
* 1 0 * 0 * *
* 1 0 * 1 * *
* 1 1 * 0 * *
* 1 1 * 1 * *
```

GA and K-Armed Bandit Problem

Competing schemata

```
* 0 0 * 0 * *
* 0 0 * 1 * *
* 0 1 * 0 * *
* 0 1 * 1 * *
* 1 0 * 0 * *
* 1 0 * 1 * *
* 1 1 * 0 * *
* 1 1 * 1 * *
```

- Since these schemata are defined over the same positions, they compete with one another for precious population slots.

GA and K-Armed Bandit Problem

Competing schemata

```
* 0 0 * 0 * *
* 0 0 * 1 * *
* 0 1 * 0 * *
* 0 1 * 1 * *
* 1 0 * 0 * *
* 1 0 * 1 * *
* 1 1 * 0 * *
* 1 1 * 1 * *
```

- Since these schemata are defined over the same positions, they compete with one another for precious population slots.
- Each schema will grow or decay exponentially based on its average fitness.

GA and K-Armed Bandit Problem

Competing schemata

*	0	0	*	0	*	*
*	0	0	*	1	*	*
*	0	1	*	0	*	*
*	0	1	*	1	*	*
*	1	0	*	0	*	*
*	1	0	*	1	*	*
*	1	1	*	0	*	*
*	1	1	*	1	*	*

- Since these schemata are defined over the same positions, they compete with one another for precious population slots.
- Each schema will grow or decay exponentially based on its average fitness.
- The difference here we assumed only 8 schemata over three positions (i.e., 8-arm bandit)

GA and K-Armed Bandit Problem

Competing schemata

```
* 0 0 * 0 * *
* 0 0 * 1 * *
* 0 1 * 0 * *
* 0 1 * 1 * *
* 1 0 * 0 * *
* 1 0 * 1 * *
* 1 1 * 0 * *
* 1 1 * 1 * *
```

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- In GA we have a number of problems proceeding in parallel

GA and K-Armed Bandit Problem

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- Not all problems are played equally due to the destructive effect of crossover and mutation

Outline

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- Holland suggested with a particular defining length, we can estimate a lower bound on the number of unique schemata processed by an initially random population of strings.

How many schema are processed usefully? $O(n^3)$ proof

- Lets start by counting the number of schemata of defining length $\delta(H)$ that cover a single string in the population.

e.g. $\ell = 10$ and $\delta(H) = 4$

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$$n_s = n.2^{\delta(H)-1}.(\ell - \delta(H) + 1)$$
- Holland assumed initial population size $n = 2^{\delta(H)/2}$ or $n^2 = 2^{\delta(H)}$, then
$$n_s = n^3.\frac{\ell - \delta(H) + 1}{2} = O(n^3)$$

Criticize on Holland n^3 argument

- The estimate for n_s depends upon a particular choice of population size and any deviation in population size invalidates the derivation.
- Increase in population size decreases the exponent, thereby decreasing the apparent leverage.

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- The probability of one or more successes is then given by

$$p\left(\texttt{at least one order j success in size m population}\right) = 1 - \left[1 - \left(\frac{1}{2}\right)^{j}\right]^{m}$$

New Estimate

- There are $C(\ell, j)$ different ways to select the j positions in a string of length ℓ. Moreover, j fixed positions there are 2^j different schemata (a 0 or 1 at any of the j positions) thus expect to have the following number of schemata with one or more representatives in a population of size m:

$$\binom{\ell}{j} 2^j \left[1 - \left[1 - \left(\frac{1}{2}\right)^j\right]^m\right]$$

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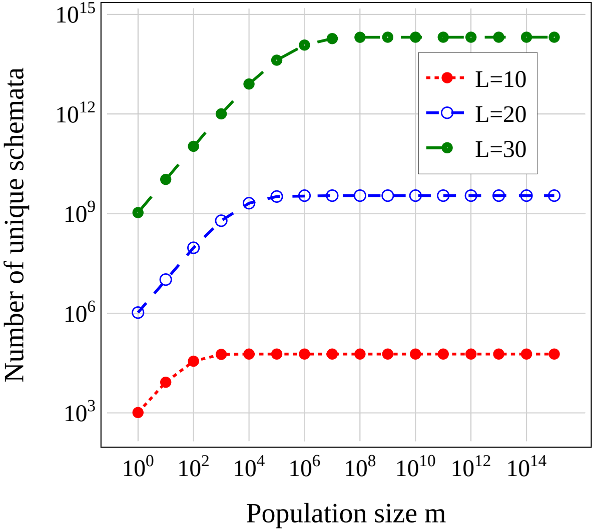
- No closed form formula or asyeptotic relation has been discovered for this expression

New Estimate



Outline

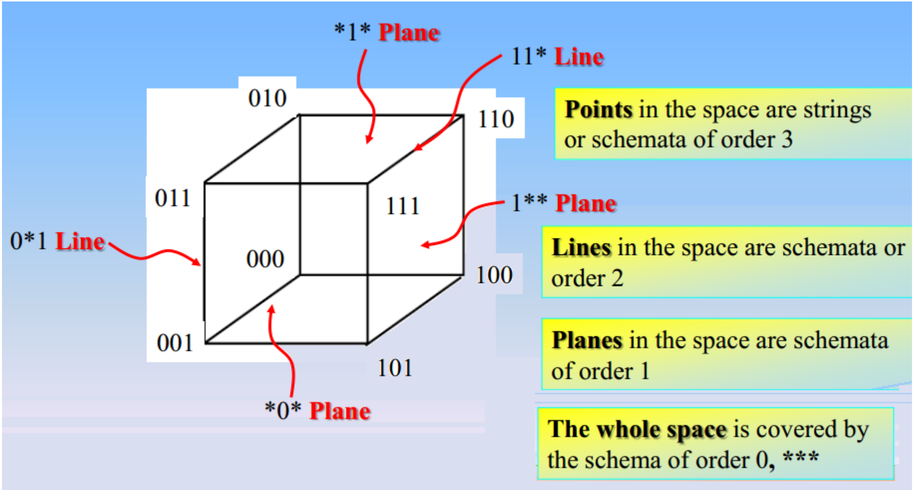
Hyperplane

Hyperplane

A hyperplane is a concept in geometry. It is a generalization of the concept of a plane.

- In 1-D space (such as a line), a hyperplane is a point; it divides a line into two rays.
- In 2-D space (such as the xy plane), a hyperplane is a line; it divides the plane into two half-planes.
- In 3-D space a hyperplane is an ordinary plane; it divides the space into two half-spaces.
- This concept can also be applied to four-dimensional space and beyond, where the dividing object is simply referred to as a hyperplane

Visualization of Schemata as Hyperplanes in 3-D Space

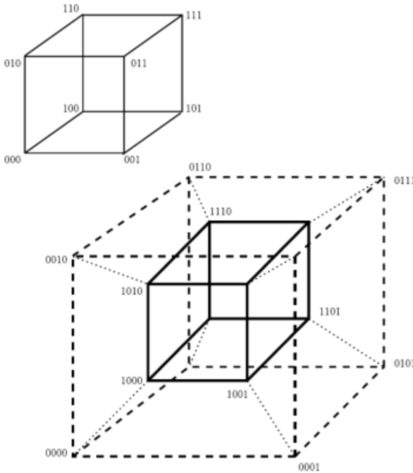


We can think of a GA cutting across different hyperplanes to search for improved performance.

Schemeta as Hyperplane

All adjacent corners are labeled by bit strings that differ by exactly 1bit
This creates an assignment to the points in hyperspace that gives the proper adjacency in the space between strings that are 1 bit different.

- inner cube: corresponds to 1***
- outer cube corresponds to 0***
- fronts of both cubes: *0**
- front of the inner cube: order-2 hyperplane 10**



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Questions